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Directed Integration.

By H. B. Phillips.

In case of an integral along a curve,

$$\int P dx + Q dy = \text{Lim}\ \Sigma P \Delta x + Q \Delta y,$$

the increments Δx and Δy may be positive or negative according as x and y are increasing or decreasing. In case of a double integral, however,

$$\int\int f(x, y)\, dx dy = \text{Lim}\ \Sigma\Sigma f(x, y)\, \Delta x \Delta y,$$

the element $\Delta x \Delta y$ is usually considered positive, or at least, invariable in sign. This introduces difficulties similar to those which occur when we attempt to banish the minus sign from algebra or analytic geometry. For instance, in a change of variable, it is necessary to assume that the Jacobian has an invariable sign.

Physicists avoid these difficulties by introducing a cosine which is positive or negative as required. Mathematicians accomplish the same result by making a change of variable, thus obtaining an element of integration which need not change sign. I wish to show in this paper how the algebraic sign can be directly attached to the element of integration, multiple integrals being treated in this respect like curvilinear integrals. The equations for change of variable and those connecting line, surface, and volume integrals, present themselves much more naturally in this form. In this discussion I shall not enter into questions of existence and convergence. These matters are treated in practically the same way whether the integral is directed or not. I shall also consider only two and three dimensions, although the extension to higher spaces is immediate.

Directed Regions.—A surface is called one-sided if it is possible to pass from a point on one side of the surface to a point on the other without passing through the surface or across its border. If this is not possible, the surface is called two-sided. A simple one-sided surface can be formed by twisting a strip of paper through 180° and bringing its ends together.

If a surface is two-sided, one side can be considered positive, the other negative. We shall assign a direction or sense to a region on a two-sided surface by assigning a direction around its border. In a right-hand system the positive direction is usually chosen such that an observer on the positive side of the surface finds the region on his left when he moves in the positive direction along the border. It should be noted that this direction does not belong to the border, but to the region of which it is the border. Thus a given direction around a great circle of a sphere is positive for the hemisphere on one side, and negative for that on the other.

Surface Integrals.—Let x and y be one-valued and continuous functions defined at each point of a region Γ on a plane or two-sided surface. Divide Γ into elementary regions, or cells, by two sets of curves

$$x = \text{constant}, \quad y = \text{constant}.$$

Any one of these cells whose boundary is a simple quadrilateral with two pairs of opposite sides belonging to the curves x, $x+\Delta x$ and y, $y+\Delta y$ will be called regular. Irregular cells may be bounded by less than four curves, or by four curves that are not of this simple type.

In the definition of the integral only the regular cells will be used. Hence it is assumed that the irregular cells can be enclosed in a region or set of regions whose total area approaches zero when Δx and Δy approach zero. This is certainly true of the curves ordinarily used in integration. It would not be true if the two systems of curves $x=$const. and $y=$const. were the same.

Choose a direction around one of the quadrilaterals. Then for that quadrilateral we define $\Delta x \Delta y$ as the product obtained by multiplying the increments of x and y which are found by passing around the quadrilateral in the chosen direction so as first to traverse a curve $y=$const., and then a curve $x=$const. The sign of $\Delta x \Delta y$ is fixed for a given quadrilateral and a given direction around it. Thus in the quadrilateral $ABCD$, if AB and DC are portions of curves $y=$const., and AD and BC portions of curves $x=$const., we may take Δx from A to B and Δy from B to C, or we may take Δx from C to D and Δy from D to A. In the second case the signs of Δx and Δy are both changed, and so $\Delta x \Delta y$ is not changed. Similarly, the product $\Delta y \Delta x$ is obtained by traversing the quadrilateral in the same direction, first traversing a curve $x=$const. and then a curve $y=$const. Inspection of a figure will make it clear that one of the increments in $\Delta y \Delta x$ differs in sign from the corresponding increment in $\Delta x \Delta y$. Hence

$$\Delta y \Delta x = -\Delta x \Delta y. \tag{1}$$

Let the same direction be taken around all the quadrilaterals into which Γ is divided. Then, if (x, y) is any point in the quadrilateral to which $\Delta x \Delta y$ belongs, we define the integral of $f(x, y)$,

$$\int\int f(x, y)\, dxdy,$$

in the chosen direction over Γ as the limit (if it exists) approached by the sum

$$\Sigma\Sigma f(x, y)\, \Delta x \Delta y$$

when Δx and Δy approach zero, the summation being for all the regular cells within Γ. Similarly,

$$\int\int f(x, y)\, dydx = \text{Lim}\ \Sigma\Sigma f(x, y)\, \Delta y \Delta x.$$

Therefore, by equation (1),

$$\int\int f(x, y)\, dydx = -\int\int f(x, y)\, dxdy. \tag{2}$$

It should be noted that the order in which the differentials are written does not indicate an order of integration. In fact, no order of integration is considered. The integrals are multiple, not iterated.

We have assumed that Δx is determined along the curves $y=$const., and Δy along the curves $x=$const. It is a very important fact that one of these increments could be determined along a third set of curves $w=$const. Thus, if we resolve Γ into cells by the curves $x=$const. and $w=$const., and in each quadrilateral determine Δy on $x=$const. as before, but Δx on $w=$const., the value of the integral will not be changed, provided the total area of the irregular cells formed by the new curves has a zero limit. For Γ can be resolved into strips between consecutive curves x, $x+\Delta x$. All the quadrilaterals in a strip have the same Δx. Also Δy is taken in both cases along the curves $x=$const. Hence, in the change assumed, the part of the summation belonging to this strip is affected only through the change in the distribution of the intervals Δy. This does not affect the limit.

Volume Integrals.—Triple integrals are defined in a similar way. Let x, y, z be one-valued and continuous functions defined at each point of a region Γ of space. Divide Γ into cells by means of three sets of surfaces $x=$const., $y=$const., and $z=$const. We shall call the cells regular which are bounded on opposite sides by three pairs of surfaces x and $x+\Delta x$, y and $y+\Delta y$, z and $z+\Delta z$. We assume that when Δx, Δy, and Δz approach zero, the total volume of the irregular cells approaches zero.

Let AB, BC, CD be consecutive edges of a cell, y and z being constant on AB, z and x on BC, x and y on CD. Let the outer surface of the cell be considered positive. The path BCD determines a direction (positive or negative) about the face of the cell in which B, C, D lie. I call this the direction, or sense, of the cell $ABCD$.

For a given cell taken with an assigned direction, we define $\Delta x \Delta y \Delta z$ as the product obtained by multiplying the increments of x, y, z which are found by passing along consecutive edges of the cell in the assigned direction, first traversing an edge on which x alone varies, then one on which y alone varies, and finally one on which z alone varies. It is easy to verify that this fixes $\Delta x \Delta y \Delta z$ not only in magnitude, but also in sign when the cell is given and a direction assigned to it. Thus let the cell be a rectangular box $ABCDEFG$ formed by the parallel rectangles $ABCH$ and $FEDG$ (x varying on AB, y on BC, and z on CD). We may take Δx on AB, Δy on BC, and Δz on CB, or we may take Δx on EF, Δy on FG, and Δz on GH. In the second case two signs (those of Δx and Δz) are changed, and so $\Delta x \Delta y \Delta z$ is not changed.

Similarly, the product $\Delta y \Delta z \Delta x$ is obtained by traversing the edges of the cell in the same direction, first traversing an edge on which y alone varies, then one on which z alone varies, and finally, one on which x alone varies. The other products of Δx, Δy, Δz are defined in a similar way. It is easy to verify that each inversion of the order of Δx, Δy, Δz introduces a negative sign in the result. Thus

$$\Delta x \Delta y \Delta z = -\Delta x \Delta z \Delta y = \Delta z \Delta x \Delta y. \tag{3}$$

Let all the cells in a region Γ be taken in the same direction. If (x, y, z) is any point within the cell to which $\Delta x \Delta y \Delta z$ refers, the integral in the assigned direction over Γ,

$$\int\int\int f(x, y, z)\, dx dy dz,$$

is defined as the limit (if it exists) approached by

$$\Sigma\Sigma\Sigma f(x, y, z) \Delta x \Delta y \Delta z$$

when Δx, Δy, and Δz approach zero, the summation being for all the regular cells within Γ. Similarly,

$$\int\int\int f(x, y, z)\, dy dz dx = \text{Lim}\ \Sigma\Sigma\Sigma f(x, y, z) \Delta y \Delta z \Delta x,$$

etc. Hence, from (3),

$$\int\int\int f(x, y, z)\, dx dy dz = -\int\int\int f(x, y, z)\, dx dz dy = \int\int\int f(x, y, z)\, dz dx dy. \tag{4}$$

As in case of double integration, one of the sets of surfaces $x = \text{const.}$, $y = \text{const.}$, $z = \text{const.}$, can be replaced by a third set of surfaces, provided the irregular cells thus introduced have a total volume that approaches zero in the limit.

Expression of an Integral in Terms of Boundary Values.—A simple integral is expressed in terms of its limits by the formula

$$\int_a^b du = u\Big|_a^b = u(b) - u(a). \tag{5}$$

Similar formulas apply to multiple integrals.

Let u be a function of x and y. In the integral,

$$\int\int du dx,$$

let du be taken along $x=$const. We may take dx along $u=$const. or $y=$const. as we choose. The integral of du along a curve $x=$const. from one intersection with the border to another is given by equation (5). Hence, if we evaluate the integral by summing first with respect to u and then with respect to x, we get

$$\int\int du dx = \int u dx. \tag{6}$$

The double integral is taken over a region Γ, the simple integral over the boundary of Γ. Since du and dx occur on consecutive sides of a quadrilateral in the double integral, the direction of integration around the boundary must be such that if BC belongs to the boundary and $ABCD$ is a quadrilateral directed as in the double integral, then the integral along the boundary is in the direction BC.

Similarly, if u is a function of x, y, z,

$$\int\int\int du dx dy = \int\int u dx dy. \tag{7}$$

The triple integral is taken over a region Γ, the double integral over its boundary. The directions of integration are so related that if $ABCD$ is a cell of the triple integral with face BCD in the boundary, then BCD gives in that face the direction of the double integral.

Illustrations of these formulas are furnished by the theorems of Green, Stokes, and Gauss. Suppose, for example, P, Q, R are functions of x, y, z on a two-sided surface. Then

$$\int P dx = \int\int dP dx,$$

the two integrals being taken over a region of the surface and around its boundary, respectively. Since dP is determined on the curves $x=$const.,

$$dP = \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz,$$

and

$$\int P dx = \int\int \frac{\partial P}{\partial y} dy dx + \frac{\partial P}{\partial z} dz dx.$$

In these integrals dy and dz are taken along $x=\text{const.}$ In the combination $dydx$ we may take dx along $y=\text{const.}$ and in $dzdx$ along $z=\text{const.}$ Similarly,

$$\int Qdy=\int\int\frac{\partial Q}{\partial x}dxdy+\frac{\partial Q}{\partial z}dzdy,$$

$$\int Rdz=\int\int\frac{\partial R}{\partial x}dxdz+\frac{\partial R}{\partial y}dydz.$$

Adding these equations and changing the sign each time we invert the order of differentials, we get

$$\int Pdx+Qdy+Rdz=\int\int\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy+\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dzdx,$$

which is Stokes' Theorem. Since

$$dP=\frac{\partial P}{\partial x}dx+\frac{\partial P}{\partial y}dy+\frac{\partial P}{\partial z}dz,$$

the equation

$$\int\int dPdx=\int\int\frac{\partial P}{\partial y}dydx+\frac{\partial P}{\partial z}dzdx,$$

suggests that symbolically

$$\int\int\frac{\partial P}{\partial x}dxdx=0. \tag{8}$$

This is in line with our definition since one set of curves does not give rise to any regular quadrilaterals, and so the summation from which the integral might be defined is zero.

Change of Variable.—Let x, y be functions of u, v. Then

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv$$

along any curve. Hence

$$\int\int fdxdy=\int\int f\frac{\partial x}{\partial u}dudy+\int\int f\frac{\partial x}{\partial v}dvdy.$$

In these integrals dy is determined along the curves $x=\text{const.}$ Since this is not significant we may determine dy in the first integral on the curves $v=\text{const.}$, and in the second integral on the curves $u=\text{const.}$ In the first case

$$dy=\frac{\partial y}{\partial v}dv,$$

and in the second

$$dy=\frac{\partial y}{\partial u}du,$$

whence

$$\int\int fdxdy=\int\int f\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}dudv+\int\int f\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}dvdu=\int\int f\left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right]dudv. \tag{9}$$

This result could be obtained by using the values

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv,$$

$$dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv,$$

in $\int\int f dxdy$, expanding, and assuming that the integrals containing $dudu$ and $dvdv$ are zero.

In a similar way we show

$$\int\int\int f dxdydz=\int\int\int f \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} dudvdw. \tag{10}$$

Equations (9) and (10) are valid whether the Jacobian has an invariable sign or not. Irregular cells will, however, usually occur in the neighborhood of a point where the Jacobian is zero. It may therefore be necessary that it be possible to enclose such points in an area in case of a double integral and in a volume in case of a triple integral which has a limit zero.

While an integral is represented by a number, the summation process upon which the integral is based is as much geometric as arithmetic. Symbolically the integral is a function in which $dxdy$ is equivalent to $-dydx$ and dx^2 to zero. This can be expressed by means of vectors. Yet dx, dy, dz are not vectors, and integration belongs no more to vector analysis than algebra does. They both belong to the larger field of quantities having sign, but not direction.